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Review of Predicate Logic

- Besides keeping the connectives from PL, Predicate Logic (PrL) decomposes simple statements into smaller parts: predicates, terms and quantifiers.

(0) John is tall.
T(j)

(1) John is taller than Bill.
T(j,b)

(2) Everybody sleeps.
$\forall$x [S(x)]

(3) Somebody likes David.
$\exists$x [L(x,d)]

1. Syntax of PrL.

- Primitive vocabulary:

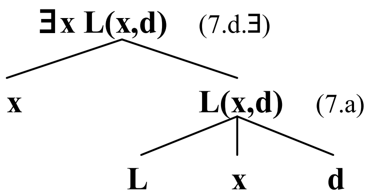
(4) Lexical entries, with a denotation of their own:
a. A set of individual constants, represented with the letters **a**, **b**, **c**, **d**…
b. A set of individual variables $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$,… $\mathbf{y_0}$, $\mathbf{y_1}$, $\mathbf{y_2}$,… Individual constants and individual variables together constitute the set of terms.
c. A set of predicates, each with a fixed n-arity, represented by **P**, **Q**, **R** …

(5) Symbols treated syncategorematically:
a.The PL logical connectives.
b. The quantifier symbols $\exists$ and $\forall$.

- Syntactic rules:

(6) a. If P is a n-ary predicate and $t_1 \ldots t_n$ are all terms, then $P(t_1 \ldots t_n)$ is an atomic formula.
b. If ϕ is a formula, then $\neg\phi$ is a formula.
c. If ϕ and ψ are formulae, then $(\phi\wedge\psi)$, $(\phi\vee\psi)$, $(\phi\rightarrow\psi)$, $(\phi\leftrightarrow\psi)$ are formulae too.
d. If ϕ is a formula and v is a variable, then $\forall v\phi$, $\exists v\phi$ are formulae too.
e. Nothing else is a formula in PrL.

(7)



QUESTION 1: Draw the syntactic tree for the expressions in (8) that are well-formed formulae of PrL.

(8) a. $\exists\forall(Qa \rightarrow PR(b)(c))$
b. $\forall x\ (P(x) \rightarrow \exists y\ Q(x,y))$
c. $\exists x_1\ \forall x_2\ (\ P(x_1,x_2) \rightarrow (R(x_1) \wedge Q(x_2,a))\ )$

QUESTION 2: Translate into PrL the following English sentences: [Mostly from GAMUT]

(9) a. John likes Susan.

b. John has a cat that he spoils.

c. Everything is bitter or sweet.

d. Either everything is bitter or everything is sweet.

e. There is something that everybody told Mary.

f. Everybody told Mary something.

g. If all logicians are smart, then Alfred is smart too.

h. Nobody came.

i. A whale is a mammal.

■ Some syntactic notions:

(10) If x is any variable and φ is a formula to which a quantifier has been attached by rule (7.d) to produce $\forall x\varphi$ or $\exists x\varphi$, then we say that φ is the *scope* of the attached quantifier and that φ or any part of φ *lies in the scope* of that quantifier.

(11) An occurrence of a variable x is *bound* if it occurs in the scope of $\forall x$ or $\exists x$. A variable is *free* if it is not bound.

(12) Formulae with no free variables are called closed formulae, formulae (simpliciter) or sentences.
Formulae containing a free variable are called open formulae or propositional functions.

2. Semantics of PrL.

(13) Variable assignments: g: set of variables → universe of individuals, D_e

$[\![\ ]\!]^{s,g}$

(14) a. If α is a constant (excluding syncategorematically treated symbols), then $[\![\alpha]\!]^{s,g}$ is specified in the Lexikon for each s.

b. If α is a variable, then $[\![\alpha]\!]^{s,g} = g(\alpha)$

(15) $g^{d/v}$ reads as "the variable assignment g' that is exactly like g except (maybe) for g(v), which equals the individual d".

(16) QUESTION: Complete the equivalences:

$g(\mathbf{x})$ = Mary	$g^{Paul/\mathbf{x}}(\mathbf{x})$=	$g^{Paul/\mathbf{x}\ Susan/\mathbf{x}}(\mathbf{x})$=	$g^{Paul/\mathbf{x}\ Susan/\mathbf{y}}(\mathbf{x})$=
$g(\mathbf{y})$ = Susan	$g^{Paul/\mathbf{x}}(\mathbf{y})$=	$g^{Paul/\mathbf{x}\ Susan/\mathbf{x}}(\mathbf{y})$=	$g^{Paul/\mathbf{x}\ Susan/\mathbf{y}}(\mathbf{y})$=

(17) a. If P is a n-ary predicate and $t_1 \ldots t_n$ are all terms, then, for any s,

$[\![P(t_1 \ldots t_n)]\!]^{s,g} = 1$ iff $<[\![t_1]\!]^{s,g}, \ldots, [\![t_n]\!]^{s,g}> \in [\![P]\!]^{s,g}$

If φ and ψ are formulae, then, for any situation s,

b. $[\![\neg\phi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g} = 0$

c. $[\![\phi\wedge\psi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g} = 1$ and $[\![\psi]\!]^{s,g} = 1$

$[\![\phi\vee\psi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g} = 1$ or $[\![\psi]\!]^{s,g} = 1$

$[\![\phi\rightarrow\psi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g} = 0$ or $[\![\psi]\!]^{s,g} = 1$

$[\![\phi\leftrightarrow\psi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g} = [\![\psi]\!]^{s,g}$

d. If φ is a formula and v is a variable, then, for any situation s,

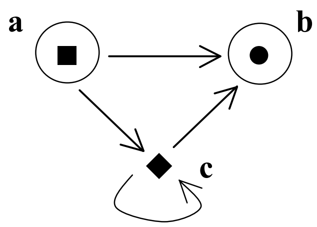
$[\![\forall v\phi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g^{d/v}} = 1$ for all the $d \in D_e$.

$[\![\exists v\phi]\!]^{s,g} = 1$ iff $[\![\phi]\!]^{s,g^{d/v}} = 1$ for some $d \in D_e$.

(18) For any formula ϕ, $[\![\phi]\!]^{s} = 1$ iff, for all assignments g, $[\![\phi]\!]^{s,g} = 1$.

QUESTION 5: Let us take the situation s depicted in (19). Let us take a language PrL_1 such that: the constants **a**, **b**, and **c** denote the individuals ■, ● and ◆, respectively, the unary predicate **A** denotes the set of individuals with a circle around, and the binary predicate **R** denotes the relation encoded by the arrows (**R(x,y)** is true iff there is an arrow from x to y). [≈GAMUT]

(19)



Determine the truth value of the following formulae of PrL_1 in s, justifying it in detail.

(20) a. $\forall x\ (A(x) \rightarrow \exists y\ (R(x,y)))$

b. $\exists x \exists y\ (R(x,y) \wedge \neg R(y,x) \wedge \exists z\ (R(x,z) \wedge R(z,y)))$

c. $\exists x \exists y \exists z \exists u\ (R(z,x) \wedge R(u,y) \wedge A(z) \wedge \neg A(u))$

3. Some equivalences [From Partee *et al.*]

For any predicate π and formula ϕ:

(21) Law of Quantifier Negation:

$\neg\forall x\,(\pi(x)) \quad\Leftrightarrow\quad \exists x\,(\neg\pi(x))$

[And, by $\neg\neg\,\phi \Leftrightarrow \phi$, also:

$\forall x\,(\pi(x)) \quad\Leftrightarrow\quad \neg\exists x\,(\neg\pi(x))$

$\neg\forall x\,(\neg\pi(x)) \quad\Leftrightarrow\quad \exists x\,(\pi(x))$

$\forall x\,(\neg\pi(x)) \quad\Leftrightarrow\quad \neg\exists x\,(\pi(x))$]

(22) Laws of Quantifier (In)Dependence:

a. $\forall x\forall y\,(\pi(x,y)) \quad\Leftrightarrow\quad \forall y\forall x\,(\pi(x,y))$

b. $\exists x\exists y\,(\pi(x,y)) \quad\Leftrightarrow\quad \exists y\exists x\,(\pi(x,y))$

c. $\exists x\forall y\,(\pi(x,y)) \quad\Rightarrow\quad \forall y\exists x\,(\pi(x,y))$

(23) Laws of Quantifier Distribution:

a. $\forall x\;(\pi(x) \wedge \rho(x)) \quad\Leftrightarrow\quad \forall x\,(\pi(x)) \;\wedge\; \forall x\,(\rho(x))$

b. $\exists x\;(\pi(x) \vee \rho(x)) \quad\Leftrightarrow\quad \exists x\,(\pi(x)) \;\vee\; \exists x\,(\rho(x))$

c. $\forall x\,(\pi(x)) \;\vee\; \forall x\,(\rho(x)) \quad\Rightarrow\quad \forall x\;(\pi(x) \vee \rho(x))$

d. $\exists x\;(\pi(x) \wedge \rho(x)) \quad\Rightarrow\quad \exists x\,(\pi(x)) \;\wedge\; \exists x\,(\rho(x))$

(24) Laws of Quantifier Movement:

a. $\phi \rightarrow \forall x(\pi(x)) \quad\Leftrightarrow\quad \forall x\,(\phi \rightarrow \pi(x))$

provided that x is not free in ϕ.

b. $\phi \rightarrow \exists x(\pi(x)) \quad\Leftrightarrow\quad \exists x\,(\phi \rightarrow \pi(x))$

provided that x is not free in ϕ and that someone exists.

c. $\forall x(\pi(x)) \rightarrow \phi \quad\Leftrightarrow\quad \exists x\;(\pi(x) \rightarrow \phi)$

provided that x is not free in ϕ and that someone exists.

d. $\exists x(\pi(x)) \rightarrow \phi \quad\Leftrightarrow\quad \forall x\;(\pi(x) \rightarrow \phi)$

provided that x is not free in ϕ.

(25) Example:

$[\exists x(P(x)) \rightarrow \forall y(Q(y))] \quad\Leftrightarrow\quad \forall y\;[\exists x(P(x)) \rightarrow (Q(y))]$ (24.a)

$\Leftrightarrow\quad \forall y\;\forall x\;[(P(x)) \rightarrow (Q(y))]$ (24.d)